Formulae for pricing optons on normally distributed variables

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1 General defintions and formuation

We follow standard nomenclature taken from Abramowitz & Stegun[1] and Johnson & Kotz IV [2], namely for the univariate and bivariate densities and cumulative distributions:

$$Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{1}$$

$$P(x) = \int_{-\infty}^{x} Z(x')\mathrm{d}x' \qquad Q(x) = 1 - P(x) \tag{2}$$

$$g(x,y;\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}\left(\frac{x^2+y^2-2\rho xy}{1-\rho^2}\right)\right] \tag{3}$$

$$L(a,b;\rho) = \int_{h}^{\infty}\int_{k}^{\infty} g(x,y;\rho)\mathrm{d}x\mathrm{d}y \tag{4}$$

$$\Phi(a,b;\rho) = \int_{-\infty}^{h}\int_{-\infty}^{k} g(x,y;\rho)\mathrm{d}x\mathrm{d}y \tag{5}$$

$$Z^{(m)}(x) = \frac{\mathrm{d}^m Z(x)}{\mathrm{d}x^m} \tag{6}$$

$$He_n(x) = (-1)^n \frac{Z^{(m)}(x)}{Z(x)} \tag{7}$$

A useful identity[1, eqn 26.3.2] is

$$g(x,y;\rho) = \frac{Z(x)}{\sqrt{1-\rho^2}} Z\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right) \tag{8}$$

We will also need (via integration by parts)

$$\int_{-\infty}^{\infty} Z^{(m)}(x)(x-K)^{+}\mathrm{d}x = Z^{(m-2)}(K) \tag{9}$$

2 Valuation and Greeks

We value call options of the form:

$$C(F, X, \sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (S - X)^{+} \exp\left[-\frac{1}{2}\left(\frac{S-F}{\sigma}\right)^{2}\right] \mathrm{d}S \qquad (10)$$

using $d = (F - X)/\sigma$ (without subscript *pace* Black-Scholes), simple straightforward evaluations yield:

$$\begin{aligned} C &= \sigma\left[dP(d) + Z(d)\right] \\ C_F &= P(d) \\ C_{FF} &= \frac{Z(d)}{\sigma} \\ C_\sigma &= Z(d) \\ C_{\sigma\sigma} &= d^2 \frac{Z(d)}{\sigma} \\ C_{F\sigma} &= -d\frac{Z(d)}{\sigma} \end{aligned} \qquad (11)$$

References

[1] Abramowitz Milton; Stegun, Irene Ann, eds. (1983) [June 1964]. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.*; Dover Publications.

[2] Johnson, N. L.; Kotz, S. (1972) *Distributions in statistics: Continuous Multivariate Distributions* ; Wiley

[3] Grimmett,G..; Welsh, D. (1986) *Probability: an introduction* ; OUP

[4] Austing, P.; (2014) *Smile Pricing Explained* ; Palgrave

[5] Haug