Useful formulas for Black76

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Using standard notations, namely for the expression of Greeks writing the partial dervative of F with respect to x as the subscript $\frac{\partial F}{\partial x} = F_x$ but using standard variables names for the Black76 functions (which also contain subscripts: hopefully without confusion)

$$C(F, X, \tau) = \exp(-r\tau)(FN(d_1) - XN(d_2)) \tag{1}$$

$$P(F, X, \tau) = \exp(-r\tau)(XN(-d_2) - FN(-d_1)) \tag{2}$$

$$d_1 = \frac{\ln F/X + \sigma^2\tau/2}{\sigma\sqrt{\tau}} \quad d_2 = \frac{\ln F/X - \sigma^2\tau/2}{\sigma\sqrt{\tau}} \tag{3}$$

$$\begin{aligned}
C_F &= \exp(-r\tau)N(d_1) \\
C_X &= -\exp(-r\tau)N(d_2) \\
C_{FF} &= \exp(-r\tau)\frac{n(d_1)}{F\sigma\sqrt{\tau}} \\
C_{FF} &= \frac{C_\sigma}{F^2\sigma\tau} \\
C_\sigma &= \exp(-r\tau)F\sqrt{\tau}n(d_1) \\
C_\tau &= -rC + \frac{1}{2}\sigma^2F^2C_{FF} \quad \text{Black76 PDE} \\
C_\tau &= -rC + \frac{\exp(-r\tau)n(d_1)}{2\sqrt{\tau}} \\
C_{\sigma\sigma} &= C_\sigma d_1 d_2/\sigma \quad \text{“volga”} \\
C_{F\sigma} &= -C_\sigma d_2/(F\sigma\sqrt{\tau}) \quad \text{“vanna”} \\
C_{FF\sigma} &= \frac{C_\sigma}{(F\sigma\sqrt{\tau})^2)}(d_2^2 + \sigma\sqrt{\tau}d_2 - 1) \\
C_{FFF} &= -\frac{C_\sigma}{F^3\sigma\tau}\left(\frac{d_2}{\sigma\sqrt{\tau}} + 1\right) \\
C_{FFFF} &= -\frac{C_\sigma}{F^4\sigma\tau(\sigma\sqrt{\tau})^2}(d_2^2 + 5d_2\sigma\sqrt{\tau} + 2(\sigma\sqrt{\tau})^2 - 1)
\end{aligned} \tag{4}$$

Inter-greek relations

$$\begin{aligned}
C_\sigma &= F^2\sigma\tau C_{FF} \\
C &= FC_F + XC_X \\
F^2 C_{FF} &= X^2 C_{XX} \\
C_{FX} &= -\frac{X}{F}C_{XX} = -\frac{F}{X}C_{FF}
\end{aligned} \tag{5}$$

Put-call relations

$$\begin{aligned}
C(F,X) - P(F,X) &= \exp(-r\tau)(F-X) \quad \text{Put-call parity} \\
C(F,X) &= P(X,F) \quad \text{Put-call symmetry}
\end{aligned} \tag{6}$$

Probability densities

$$\begin{aligned}
p(F(T)|F(0)) &= \frac{1}{\sqrt{2\pi}}\frac{1}{F(T)\sigma\sqrt{\tau}}\exp\left[-d_1^2/2\right] \\
&= \exp(r\tau)C_{XX}(X = F(T), F = F(0)) \\
&= \exp(r\tau)C_{FF}(F = F(T), X = F(0))
\end{aligned}$$

Breeden & Litzenberger (7)

Moments

$$\begin{aligned}
E[F^\alpha] &= \int_0^\infty \mathrm{d}F(T)F(T)^\alpha \frac{1}{\sqrt{2\pi}}\frac{1}{F(T)\sigma\sqrt{\tau}}\exp\left[-\frac{1}{2}\left(\frac{\ln F(T)/F(0) + \sigma^2\tau/2}{\sigma\sqrt{\tau}}\right)^2\right] \\
&= F(0)^\alpha e^{\sigma^2\tau(\alpha^2-\alpha)/2}
\end{aligned} \tag{8}$$