Are you the sub-optimal link?*

T. Weston
tom@eponymous.org.uk

W. Weston
w.weston@ntlworld.com

January 6, 2002

1 Introduction

Structure of paper is

1. Introduction
2. Rules of game
3. Method of iterated subgame-perfect equilibrium
4. Subproblems
5. Results & discussion

2 Subproblems

Two subproblems need to be solved, (a) game is finally settled as "head-to-head" competition between two players; five questions asked each, winner answers the greater number correctly. If draw, resolved by "sudden death"[1] (need to check the sudden death aspect agrees with how played on TV?) and (b) at each round "pot" is increased by cyclically asking surviving contestants questions, each may "bank" existing earnings or may risk adding to unbanked earnings at greater rate, risking loss of all unbanked earnings.

2.1 First subproblem

Two players are A and B. A has a probability p_A of answering correctly, B's probability is p_B. A players has probability of answering n out of 5 questions ($0 \leq n \leq 5$) correctly given by

$$\pi(n,p) = {}^5C_n p^n (1-p)^{5-n} \quad (1)$$

*working title

[1] cf. penalty shoot out in football world cup

where p is the probability of answering each individual question correctly. Therefore

$$\begin{array}{ll} \text{A wins} & \sum_i \sum_{j<i} \pi(i,p_A)\pi(j,p_B) \\ \text{B wins} & \sum_i \sum_{j>i} \pi(i,p_A)\pi(j,p_B) \\ \text{draw} & \sum_i \pi(i,p_A)\pi(i,p_B) \end{array} \tag{2}$$

To model the sudden death aspect in the case of draw, consider Bayes' formula in application to the problem:

$$\begin{aligned} Prob\begin{pmatrix}\text{A wins on}\\ \text{round } n+1\end{pmatrix} &= Prob\left(\begin{matrix}\text{A wins on}\\ \text{round } n+1\end{matrix}\middle|\begin{matrix}\text{draw on}\\ \text{round } n\end{matrix}\right) Prob\begin{pmatrix}\text{draw on}\\ \text{round } n\end{pmatrix} \\ Prob\begin{pmatrix}\text{B wins on}\\ \text{round } n+1\end{pmatrix} &= Prob\left(\begin{matrix}\text{B wins on}\\ \text{round } n+1\end{matrix}\middle|\begin{matrix}\text{draw on}\\ \text{round } n\end{matrix}\right) Prob\begin{pmatrix}\text{draw on}\\ \text{round } n\end{pmatrix} \\ Prob\begin{pmatrix}\text{draw on}\\ \text{round } n+1\end{pmatrix} &= Prob\left(\begin{matrix}\text{draw on}\\ \text{round } n+1\end{matrix}\middle|\begin{matrix}\text{draw on}\\ \text{round } n\end{matrix}\right) Prob\begin{pmatrix}\text{draw on}\\ \text{round } n\end{pmatrix} \end{aligned} \tag{3}$$

Since each round of the "sudden death" is independent we have immediately,

$$\begin{aligned} Prob\left(\begin{matrix}\text{A wins on}\\ \text{round } n+1\end{matrix}\middle|\begin{matrix}\text{draw on}\\ \text{round } n\end{matrix}\right) &= p_A(1-p_B) \\ Prob\left(\begin{matrix}\text{B wins on}\\ \text{round } n+1\end{matrix}\middle|\begin{matrix}\text{draw on}\\ \text{round } n\end{matrix}\right) &= p_B(1-p_A) \\ Prob\left(\begin{matrix}\text{draw on}\\ \text{round } n+1\end{matrix}\middle|\begin{matrix}\text{draw on}\\ \text{round } n\end{matrix}\right) &= 1-p_A-p_B+2p_Ap_B \end{aligned} \tag{4}$$

Then considering all the ways in which A and B can win

$$\begin{aligned} Prob(\text{A wins}) &= p_A(1-p_B)\sum_{m=0}^{\infty}(1-p_A-p_B+2p_Ap_B)^m \\ &= \frac{p_A(1-p_B)}{p_A+p_B-2p_Ap_B} \\ Prob(\text{B wins}) &= p_B(1-p_A)\sum_{m=0}^{\infty}(1-p_A-p_B+2p_Ap_B)^m \\ &= \frac{p_B(1-p_A)}{p_A+p_B-2p_Ap_B} \end{aligned} \tag{5}$$

Hence, over all the probability of A winning is

$$\sum_i \sum_{j<i} \pi(i,p_A)\pi(j,p_B) + \frac{p_A(1-p_B)}{p_A+p_B-2p_Ap_B}\sum_i \pi(i,p_A)\pi(i,p_B) \tag{6}$$

with the corresponding result for B.

2.2 Second subproblem

The second problem is a straight-forward example of the application of a backward induction argument.

The nature of this section of the game is that the players are ordered cyclically and are successively asked a sequence of Q questions. If the players answer n successive questions correctly they are nominally awarded prize money of x_n, where the values x run in a sequence of

$$\{x_1, x_2, \ldots, x_9\} = \{£10, £50, £100, £200, £300, £450, £600, £800, £1,000\}$$

(for later mathematical convenience we prepend the value $x_0 = 0$ to this set) however if the $n+1$th question is answered incorrectly the accumulated sum of money is lost. The mechanism to avoid losing acquired money in this way is the existence of a "bank". Before learning the content of a question a contestant may call "bank" and the current balance is put into the bank making it safe from being lost and the game resumes with the accumulated balance being set to x_0. The set of values stops at x_9, or £1,000, as the winnings from any round is capped at this value: if banked values exceed this value at any point the game terminates with total winnings of x_9.

The optimum strategy and the expected winnings from adopting this strategy can be formulated in the following manner. Let b_i be an amount banked, discretized in units of, say, £1. Let x_j be a possible current value of non-banked winnings where this value is a member of the set of such winnings described above. Additionally, let the set ν be defined as the set of ten integers,

$$\{\nu_0, \nu_1, \ldots, \nu_9\} = \{0, 10, 50, 100, 200, 300, 450, 600, 800, 1000\}$$

Now we set $V(q, b_i, x_j)$ to be a value of being at the point just before discovering the content of question q (out of Q), whilst still having the opportunity of calling "bank". Furthermore, let the N players have probabilities of correctly answering the questions of $p_1, p_2, \ldots, p_N$, where player 1 is asked the first question and hence that question q will be answered by player $q \bmod N$. This player is faced by a simple choice, should he either bank the current unbanked winnings and procede to play for the amount x_1 or procede without banking for the larger but riskier available winnings. In the former case the expected value of this course of action is

$$p_{q \bmod N} V(q+1, b_{i+\nu_j}, x_1) + (1 - p_{q \bmod N}) V(q+1, b_{i+\nu_j}, x_0) \tag{7}$$

and the latter, riskier choice, has the expected value

$$p_{q \bmod N} V(q+1, b_i, x_{j+1}) + (1 - p_{q \bmod N}) V(q+1, b_i, x_0) \tag{8}$$

We assert that the risk-neutral rational player who only wishes to maximise winnings choses the larger of these two expected values, and hence,

$$\begin{aligned} V(q, b_i, x_j) = \max \big[& p_{q \bmod N} V(q+1, b_{i+\nu_j}, x_1) + (1 - p_{q \bmod N}) V(q+1, b_{i+\nu_j}, x_0), \\ & p_{q \bmod N} V(q+1, b_i, x_{j+1}) + (1 - p_{q \bmod N}) V(q+1, b_i, x_0) \big] \end{aligned} \tag{9}$$

The final condition required to apply backwards induction to (9) is that after the final question, only banked value is taken from the round, hence

$$V(Q+1, b_i, x_j) = b_i \tag{10}$$

One final additional feature needs adding to the formalism represented by equation (9): the capping of winnings to x_9. This is represented by introducting a function F into the equation

$$\begin{aligned} V(q, b_i, x_j) = \max \big[& p_{q \bmod N} F(i + \nu_j, V(q+1, b_{i+\nu_j}, x_1)) \\ & +(1 - p_{q \bmod N}) F(i + \nu_j, V(q+1, b_{i+\nu_j}, x_0)), \\ p_{q \bmod N} V(q+1, b_i, x_{j+1}) & + (1 - p_{q \bmod N}) V(q+1, b_i, x_0)] \end{aligned} \tag{11}$$

where the function has the form

$$F(i, y) = \begin{cases} x_9 \text{ if } i \geq 1000 \\ y \text{ otherwise} \end{cases} \tag{12}$$