Approximate Pricing of APOs

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1 Introduction

Average Price Options (APOs) are the main vanilla instrument in the crude OTC options market. Analytic approximations are of considerable value for quick pricing and risk management. Many such approximations are based upon the technique of moment matching.

When used for the valuation of an APO, where the option is based on the average price taken between two forward dates, typically these models provide an effective volatility that may be substituted into a the standard Black76 option pricing formula where the expiry of the option is typically taken to be the final date of the averaging period. There are various published formulas of this type associated with the names of Kemna and Vorst, Turnbull and Wakeman and Levy [2]. However many of these formulas originated in the equity markets and therefore do not necessarily take into account relevant features of the volatility and price term structure.

2 Mathematical detail

In the simplest case we consider a single factor model of forward prices of the form

$$\frac{\mathrm{d}F(t,T)}{F(t,T)} = \sigma(t,T)\mathrm{d}W \tag{1}$$

where $F(t,T)$ is the forward curve observed at t for a range of maturities denonted by T, and $\sigma(t,T)$ is the equivalent volatility with W being a single Brownian motion. This is the model popularised by Clewlow and Stickland[1] in the energy markets.

The spot price is expressed as the forward price for immediate delivery: $S(t) = F(t,t)$. In practice we take the spot price to be the front month NYMEX or IPE contract. We can solve equation (1) for this spot price observed at future time t, given a forward curve observed at time t_0 $(t > t_0)$.

$$S(t) = F(t,t) = F(t_0,t)\exp\left[\int_{t_0}^{t}\sigma(t',t)\mathrm{d}W(t') - \frac{1}{2}\int_{t_0}^{t}\sigma^2(t',t)\mathrm{d}t'\right]. \qquad (2)$$

This quantity satisfies the following SDE

$$\frac{\mathrm{d}S}{S} = \left(\frac{\partial \ln F(t_0,t)}{\partial t} + \int_{t_0}^{t}\frac{\partial\sigma(t',t)}{\partial t}\mathrm{d}W(t') - \frac{1}{2}\int_{t_0}^{t}\frac{\partial\sigma^2(t',t)}{\partial t}\mathrm{d}t'\right)\mathrm{d}t + \sigma(t,t)\mathrm{d}W(t'). \qquad (3)$$

The form of the SDE reveals that only with a particular choice of volatility, namely an exponential, do the dynamics of $S(t)$ become Markovian.
The APO contacts we consider are based on arithmetic averages of the spot price. The call pays out

$$\left(\frac{1}{N}\sum_{i=1}^{N}S(t_i) - K\right)^{+} \qquad (4)$$

with an equivalent form for the put where t_i represent the fixing dates for the contract.
The moment matching algorithm explicitly calculates the first two moments of the distribution of the sum

$$A(t) = \sum_{i=1}^{N}S(t_i)/N \qquad (5)$$

and fits them to a generic lognormal distribution to obtain an effective volatility. Since the distribution has been mapped onto a lognormal function valuation and sensitivities can be calculated directly from the Black-Scholes/Black76 formalism.
To proceed with the moment calculation

$$\begin{aligned} E(A(t)) &= \frac{1}{N}\sum_i E\left(F(t,t_i)\exp\left[\int_t^{t_i}\sigma(t',t_i)\mathrm{d}W(t') - \frac{1}{2}\int_t^{t_i}\sigma^2(t',t_i)\mathrm{d}t'\right]\right) \\ &= \frac{1}{N}\sum_i F(t,t_i). \end{aligned} \qquad (6)$$

The second moment can be expressed as a double sum of products,

$$E(A(t)^2) = \frac{1}{N^2}\sum_{i,j}E(S(t_i)S(t_i)). \qquad (7)$$

Considering term by term expectations, we obtain

$$E(S(t_i)S(t_i)) = \\ F(t,t_i)F(t,t_j)\exp\left[-\frac{1}{2}\int_t^{t_i}\sigma^2(t',t_i)\mathrm{d}t' - \frac{1}{2}\int_t^{t_j}\sigma^2(t',t_j)\mathrm{d}t'\right]$$

$$
\begin{aligned}
&\times \quad E\left(\exp\left[\int_t^{t_i} \sigma(t',t_i)\mathrm{d}W(t') + \int_t^{t_j} \sigma(t',t_j)\mathrm{d}W(t')\right]\right) \\
&= \quad F(t,t_i)F(t,t_j)\exp\left[-\frac{1}{2}\int_t^{t_i}\sigma^2(t',t_i)\mathrm{d}t' - \frac{1}{2}\int_t^{t_j}\sigma^2(t',t_j)\mathrm{d}t'\right] \\
&\times \quad E\left(\exp\left[\int_t^{\min(t_i,t_j)} (\sigma(t',\min(t_i,t_j)) + \sigma(t',\max(t_i,t_j)))\mathrm{d}W(t')\right.\right. \\
&+ \quad \left.\left.\int_{\min(t_i,t_j)}^{\max(t_i,t_j)} \sigma(t',\max(t_i,t_j))\mathrm{d}W(t')\right]\right) \\
&= \quad F(t,t_i)F(t,t_j)\exp\left[\int_t^{\min(t_i,t_j)} \sigma(t',t_i)\sigma(t',t_j)\mathrm{d}t'\right] \qquad (8)
\end{aligned}
$$

and the second moment is

$$
E(A(t)^2) = \frac{1}{N^2}\sum_{i,j} F(t,t_i)F(t,t_j)\exp\left[\int_t^{\min(t_i,t_j)} \sigma(t',t_i)\sigma(t',t_j)\mathrm{d}t'\right]. \qquad (9)
$$

Choosing a lognormal density for the quantity $A(t)$ with generic volatility

$$
p(A) = \frac{1}{\sqrt{2\pi\sigma_{eff}^2 A^2 T_N}}\exp\left[-\frac{1}{2}\left(\frac{\ln A/A_0 + \sigma_{eff}^2 T_N/2}{\sigma_{eff}\sqrt{T_N}}\right)^2\right] \qquad (10)
$$

where T_N is the last observed fixing date. The moments of this distibution are

$$
E(A^\beta) = A_0^\beta \exp(\sigma_{eff}^2 T_N \beta(\beta-1)/2), \qquad (11)
$$

in particular

$$
\begin{aligned}
E(A) &= A_0 \\
E(A^2) &= A_0^2\exp(\sigma_{eff}^2 T_N) \qquad (12)
\end{aligned}
$$

Comparison of equations (6) and (9) with (12) directly gives the equivalent Black76 volatility.

We can consider a continuously observed version of this formula by taking $\delta t = (T_2 - T_1)/N$ and $\lim_{N\to\infty}$. In this case the sums become integrals and we are left with

$$
\begin{aligned}
E(A^2) &= \frac{1}{(T_2-T_1)^2}\int_{t_1=T_1}^{T_2}\int_{t_2=T_1}^{T_2} F(t,t_1)F(t,t_2) \\
&\times \exp\left[\int_t^{\min(t_i,t_j)} \sigma(t',t_i)\sigma(t',t_j)\mathrm{d}t'\right]\mathrm{d}t_1\mathrm{d}t_2 \qquad (13)
\end{aligned}
$$

Note that in the limit of $T_2 \simeq T_1 = T$ (13) becomes

$$
E(A^2) \simeq F^2\exp\left[\int_t^T \sigma^2(t',T)\mathrm{d}t'\right] \qquad (14)
$$

Using this result and (12), then the familiar European options root-mean-square average volatility formula is recovered,

$$\sigma_{eff} = \sqrt{\frac{1}{T-t}\int_t^T \sigma^2(t',T)\mathrm{d}t'}. \tag{15}$$

In order to compare this result to other published calculations, consider the case of flat volatility and price term structure, $F(t,T)=F$ and $\sigma(t,T)=\sigma$. In this particular market consider the contract which is based upon a continuous average between T_1 and T_2 observed at time t, $t<T_1<T_2$. Equation (13) now becomes

$$\begin{aligned} E(A^2) &= \frac{F^2}{(T_2-T_1)^2}\int_{t_1=T_1}^{T_2}\int_{t_2=T_1}^{T_2} \exp\left[\sigma(\min(t_1,t_2)-t)\right]\mathrm{d}t_1\mathrm{d}t_2 \\ &= \frac{2F^2}{\sigma^4(T_2-T_1)^2}\left(-(T_2-T_1)\sigma^2\exp[\sigma^2(T_1-t)]\right. \\ &+ \left.\exp[\sigma^2(T_2-t)]-\exp[\sigma^2(T_1-t)]\right). \end{aligned} \tag{16}$$

Extracting the effective Black76 volatility from this formula yields

$$\begin{aligned} \sigma_{eff}^2(T_2-t) &= \sigma^2(T_2-t) \\ &+ \ln\left[\frac{2}{\sigma^4(T_2-T_1)^2}\left[1-\sigma^2(T_2-T_1)\exp(-\sigma^2(T_2-T_1))\right.\right. \\ &- \left.\left.\exp(-\sigma^2(T_2-T_1))\right]\right]. \end{aligned} \tag{17}$$

Note that this formula may also be expanded as

$$\begin{aligned} \sigma_{eff}^2(T_2-t) &= \sigma^2(T_1-t)+\ln\left[\frac{2}{\sigma^4(T_2-T_1)^2}\left(-1-\sigma^2(T_2-T_1)+1+\sigma^2(T_2-T_1)\right.\right. \\ &\quad \left.\left.+\frac{\sigma^4(T_2-T_1)^2}{2}+\frac{\sigma^6(T_2-T_1)^3}{6}+\ldots\right)\right] \\ &= \sigma^2(T_1-t)+\ln\left[1+\frac{\sigma^2(T_2-T_1)}{3}+\ldots\right] \\ &\simeq \sigma^2(T_1-t)+\frac{\sigma^2(T_2-T_1)}{3}+\ldots \end{aligned} \tag{18}$$

As comparison, there is a published formula due to Turnbull & Wakeman [2, pp. 97-98, eqn 2.61 and 2.62 but note the typo] who quote an effective volatility of

$$\sigma_A = \sqrt{\frac{\ln(M_2)}{T_2}-2b_A} \tag{19}$$

where

$$M_1 = \frac{\exp(bT_2)-\exp(bT_1)}{b(T_2-T_1)}$$

$$
\begin{aligned}
M_2 &= \frac{2\exp(2b+\sigma^2)T_2}{(b+\sigma^2)(2b+\sigma^2)(T_2-T_1)^2} \\
&\quad + \frac{2\exp(2b+\sigma^2)T_1}{b(T_2-T_1)^2}\left[\frac{1}{2b+\sigma^2} - \frac{\exp(b(T_2-T_1))}{(b+\sigma^2)}\right] \qquad (20)
\end{aligned}
$$

where b is a cost of carry parameter. Taking $\lim_{b\to 0}$ produces $M_1 = 1$ and

$$
M_2 = \frac{2\exp(\sigma^2 T_2)}{\sigma^4(T_2-T_1)^2}\left[1+\exp(-\sigma^2(T_2-T_1))(-1-\sigma^2(T_2-T_1))\right] \qquad (21)
$$

from which (17) is recovered.

As a further check, take $\lim \delta t \to 0 = \lim T_2 - T_1 \to 0$. In this limit, the argument of the logarithm in equation (17) becomes (using L'Hopital)

$$
\begin{aligned}
&\lim_{\delta t\to 0}\left[\frac{2}{\sigma^4(T_2-T_1)^2}\left\{1-\sigma^2(T_2-T_1)\exp(-\sigma^2(T_2-T_1)) - \exp(-\sigma^2(T_2-T_1))\right\}\right] \\
= &\lim_{\delta t\to 0}\exp(-\sigma^2(T_2-T_1)) = 1 \qquad (22)
\end{aligned}
$$

hence $\sigma_{eff} = \sigma$. i.e. when the averaging period of the Asian is reduced to a small period the effective volatility reduces to that of the European.

3 Asians with a simple "cash volatility" term structure

Consider adding a simple term structure, with a form equivalent to a step function local volatility function.

$$
\sigma(t,T) = \begin{cases} \sigma_E & T-t > \delta t \\ \sigma_{cash} & 0 < T-t < \delta t \end{cases} \qquad (23)
$$

The implied volatility for such options under with this local volality function are simply

$$
(T-t)\sigma^2_{imp}(t,T) = \begin{cases} (T-t-\delta t)\sigma^2_E + \sigma^2_{cash}\delta t & T-t > \delta t \\ (T-t)\sigma^2_{cash}\delta & 0 < \delta t < T-t \end{cases} \qquad (24)
$$

Typically δt is taken to be one day. In this instance equation (16) becomes

$$
E(A^2) = \frac{F^2}{(T_2-T_1)^2}\int_{t_1=T_1}^{T_2}\int_{t_2=T_1}^{T_2} \exp\left[\sigma_E(\min(t_1,t_2)-t) + (\sigma_{cash}-\sigma_E)\delta t\right] \mathrm{d}t_1 \mathrm{d}t_2 \qquad (25)
$$

References

[1] Clewlow, L. and Stickland, C. (1999) *Valuing Energy Options in a One Factor Model Fitted to Forward Prices*, http://www.energyforum.net/feature/PDF/150.pdf

[2] Haug, E., G., (1997) *The Complete Guide to Option Pricing Formulas* McGraw-Hill