The Airline Allocated Seat Problem

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1 Introduction

The Airline Allocated Seat problem (AASP) is a toy problem to assess the distribution of waiting times when passengers board an aircraft with pre-allocated seat numbers but in random order. A frequent issue, which occurs often with budget airlines which encourage the use of hand luggage is that if there is a signifcant delay as they stow it in luggage racks, then the potential boarding can be dependent on the degree to which the sequence of boarding passengers is aligned with the seat numbers. We shall calculate the distrubition of waiting times in this situation.

2 Setup

AASP model assumptions:

1. We consider set of N passengers who will be sitting in allocated numbered plane seats $\{1, 2, \ldots, N\}$. Seat 1 is at the front of the plane which is also where the passengers board from
2. The passengers arrive in a random order and all board at the same time. Evidently there are $N!$ permutations of how the passengers can board.
3. As each passenger finds his allocated seat he or she waits a unit time (to stow luggage)
4. No passenger can get past another as they stow luggage.

3 Boarding time as a function of boarding permutation

We describe the particular permutation of passengers waiting to board as an ordered list. We are looking at how the random ordered list matches the ordered list of seats $(1, 2, \ldots, N)$, Consider the following example where $N = 6$

passenger order		seat order
4 5 1 6 2 3	$\rightarrow$	1 2 3 4 5 6

Clearly passengers $(2,3)$ advance down the plane together until they get to their seats but the remaining passengers till they are seated. Then passengers $(1,6)$ advance in a similar fashion, and then pause till seated, when the remaining two can finally be seated. The sequence is represented diagrammatically here

	passenger order		seat order
$t=0$	$\boxed{4\ 5}\ \boxed{1\ 6}\ \boxed{2\ 3}$	$\rightarrow$	$(1\ 2\ 3\ 4\ 5\ 6)$
$t=1$	$\boxed{4\ 5}\ \boxed{1\ 6}$	$\rightarrow$	$(1\ 2^\star\ 3^\star\ 4\ 5\ 6)$
$t=2$	$\boxed{4\ 5}$	$\rightarrow$	$(1^\star\ 2^\star\ 3^\star\ 4\ 5\ 6^\star)$
$t=3$		$\rightarrow$	$(1^\star\ 2^\star\ 3^\star\ 4^\star\ 5^\star\ 6^\star)$

where a starred suffix on the RHS indicates an occupied seat From this observation we conclude:

> For each ordered list that corresponds to a permutation of possible passenger ordering we partition the list into sublists of ascending seat number.
> Boarding time = number of ascending seat number partitions.

4 Enumeration of ascending partitions

Since, by assumption, the permutations of passenger orders are uniformly distributed, aside from an $N!$ normalisation factor the distribution of boarding times is equal to the enumeration of ascending seat number partitions.

We derive a recursive formula for the number,m, of partitions of the ordered list of ascending integers of length N, $(1,2,\ldots,N)$, into subpartitions which are ascending.

By analogy with binomial coefficients, we write $P(N,m)$ as the number of permutations of N integers with m ascending subpartitions. Clearly $1 \leq m \leq N$.

Now consider the example permutation (4 5 1 6 2 3) as described above for $N=6$. Clearly $m=3$ for this example as can be seen more clearly writing boxes around the paritions (4 5 1 6 2 3)$\equiv$ ($\boxed{4\ 5}\ \boxed{1\ 6}\ \boxed{2\ 3}$) Suppose we now consider contribution of this particular partition to the general case where $N=7$ by the addition of the integer 7 to generate seven new permutations. It can evidently be inserted in 7 positions:

$$
\begin{array}{ccll}
(4\ 5\ 1\ 6\ 2\ 3)\ (7) & \equiv & (\fbox{4 5}\ \fbox{1 6}\ \fbox{2 3 7}) & m=3 \\
(4\ 5\ 1\ 6)(7)\ (2\ 3) & \equiv & (\fbox{4 5}\ \fbox{1 6 7}\ \fbox{2 3}) & m=3 \\
(4\ 5)\ (7)\ (1\ 6\ 2\ 3) & \equiv & (\fbox{4 5 7}\ \fbox{1 6}\ \fbox{2 3}) & m=3 \\
 & & & \\
(4\ 5\ 1\ 6\ 2)\ (7)\ (3) & \equiv & (\fbox{4 5}\ \fbox{1 6}\ \fbox{2 7}\ \fbox{3}) & m=4 \\
(4\ 5\ 1)(7)\ (6\ 2\ 3) & \equiv & (\fbox{4 5}\ \fbox{1 7}\ \fbox{6}\ \fbox{2 3}) & m=4 \\
(4)(7)(5\ 1\ 2\ 3) & \equiv & (\fbox{4 7}\ \fbox{5}\ \fbox{1 6}\ \fbox{2 3}) & m=4 \\
(7)(4\ 5\ 1\ 6\ 2\ 3) & \equiv & (\fbox{7}\ \fbox{4 5}\ \fbox{1 6}\ \fbox{2 3}) & m=4
\end{array}
$$

Thus $m = 3$ paritioning in the $N = 6$ case produces $m = 3$ and $m = 4$ partitioning in the $N = 7$ case which is independent of the nature of intital $N = 6$,$m = 3$ partitioning.
Therefore we can write a recursive formula for the numbers of partitions $P(N, m)$:

$$P(N+1, m) = mP(N, m) + (N - m + 2)P(N, m - 1) \tag{1}$$

It is interesting to compare this formula with the equivalent one for binomial coefficients

$$C(N+1, m) = C(N, m) + C(N, m - 1) \tag{2}$$

and similarly we can construct an equivalent to Pascal's triangle for the numbers of subpartitions

$$
\begin{array}{lc}
N=1 & 1 \\
N=2 & 1\ 1 \\
N=3 & 1\ 4\ 1 \\
N=4 & 1\ 11\ 11\ 1 \\
N=5 & 1\ 26\ 66\ 26\ 1
\end{array}
$$

5 Moments of the boarding time

Since the quantity of interest to us is the boarding time we now calculate the mean boarding time. From the above discussion

$$E(\text{boarding time}) = \frac{1}{N!}\sum_{m=1}^{N} mP(N, m) \tag{3}$$

By symmetry of the distribution $P(N, m) = P(N, N - m)$ then we can simply evaluate the sum as

$$E(\text{boarding time}) = (N + 1)/2 \tag{4}$$